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The Hurst exponent as a tool for the description of magma field heterogeneity reflected in the geochemistry of growing crystals

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ABSTRACT:

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Trace element behaviour during crystallization of three alkali feldspar crystals of mixed origin was investigated. The first crystal (gm1) was growing under an intensive magma mixing regime in an active region of an inhomogeneous magmatic field. The second crystal (ref) was growing in a coherent region of this field and the third one (gm2) was growing under moderate progress in magma mixing, with the process being close to completion. The Hurst exponent (*H*) was used as a tool for the description of the local heterogeneities of the magma field during the mixing process. Values of *H* were calculated for compatible trace element patterns along each traverse for each crystal. The gm1 crystal is strongly zoned. The value of the Hurst exponent (*H*) for zones reflecting intensive chemical mixing varies between 0.06 and 0.47. It emphasizes strong anti-persistent behaviour of elements during crystallization. The zones that grew in a slightly contaminated felsic magma exhibit $H > 0.5$. It means that the process goes over a longer path than a random walk and shows increasing persistence in element behaviour with decreasing hybridization. Similarly, zones crystallized in magma regions compositionally located close to coherent characteristic or in active domains featuring a high homogenization (crystals gm2, ref) show higher *H* values.

Key words: Fractals; Hurst exponent; Magma mixing; Geochemistry; Persistent element behaviour; Anti-persistent element behavior.

INTRODUCTION

The objective of this paper is the presentation of a methodology, involving the application of chaos theory, which has been developed for the description of local heterogeneities of a magma field caused by the mixing process. Chaos theory is based on the assumption that a mathematical description of dynamic nonlinear systems is possible, for instance: a description of chaotic turbulences during mixing of liquids (in geology, magmas or fluids), when small alterations in the starting conditions are the reasons for significant changes in the final results of the process. Magma mixing is a nonlinear process, which leads to the creation of a geochemically heterogeneous magma field – a crystallization environment for numerous magmatic minerals.

Perugini and others (Perugini *et al*. 2002, 2003) described the hybridization process on a macro scale (textures recorded in rock) by applying the theory of geometric fractals, which reflected the self-similarity in this geological process. The box-counting technique (Mandelbrot 1982) was used. The area of non-contaminated magmas (with the primary composition) inside the mixing field was named 'coherent' by Perugini *et al*. (2002, 2003); the area of intense mixing of coherent magmas was named 'active'. The biggest geochemical variation occurs nowhere else but within the active field. The active area is the main factor responsible for variations in the geochemical composition of crystals created therein with regard to microchanges in element mobility occurring due to magma stirring (Słaby *et al*. 2007a, b, 2008; Słaby and Götze 2004). The variation in mineral composition should be a reflection of the melt's chaotic advection in a heterogeneous and active magma field (Słaby *et al*. 2007b, 2008). Both the three-dimensional distribution of mineral micro-heterogeneity resulting from hybridization and its mathematical fractal description have never previously been investigated. Nevertheless, fractals were used for the description of, for instance, diffusive delay, which resulted *inter alios* in the creation of an oscillative zonality in minerals (Perugini *et al*. 2005). This research proposes to use the Hurst coefficient for description of geochemical heterogeneity in minerals, such heterogeneity being a result of melt hybridization.

FRACTALS

Fractal theory was developed over several dozen years. It is recognised as a part of geometry but deals with the description of objects featuring structures different from those known from classic euclidean geometry − this is because, in classic geometry, the topological size of an object has an integer value. Still, a fractal object is a 'complex' geometrical shape and its fractal dimension is bigger than its topological dimension (and has irrational values).

Contemporary fractal theory is based on the work of eminent mathematicians from the turn of the 19th century (e.g. G. Cantor, P. Fatou, F. Hausdorff, D. Hilbert, G. Julia, G. Pean, W. Sierpiński). The notion of 'fractal' was introduced by Benoit Mandelbrot (1977), who has been recognised as the creator of fractal geometry. Mandelbrot observed an outstanding similarity of diagrams for prices of stocks analyzed over different time periods. Those objects were irregular to such extent that their description with the application of Euclidean geometry was impossible. Mandelbrot's next achievement was the observation that large numbers of fractals are present in nature, such as leaves, green plants, snow flakes, shorelines, etc. His further investigations stimulated the development of an entirely new field of science – fractal geometry. The word 'fractal' originates from the Latin word *fractus* and means 'broken' or 'fractional, fragmentary'. Fractal geometry enables nature to be looked at from a different point of view. What seems chaotic and without a hierarchic structure, gains a new value through the fractal approach because it can be assessed quantitatively. Fractals are objects, which cannot be described with the application of a single precise definition. They are defined most frequently on the basis of the relationship between their area or volume versus length, and hence they indicate the way they fill the space in which they are located.

All fractals have some common features (Mandelbrot 1982):

- they are described with a relatively simple recurrence relationship, and not with a mathematical formula;
- feature self-similarity − in the approximate or stochastic sense and not in the exact sense;
- have a non-trivial structure on any scale;
- their Hausdorff dimension is bigger than their topological dimension.

Therefore, the word 'fractal' is used for a set that has all or at least a majority of those features (Falconer 1997). Self-similarity is one of the key features of fractals, and this particular feature of a fractal object is used first in fractal analysis. Self-similarity is a particular case of self-affinity (Peitgen *et al.* 2002). On the other hand, the Hurst exponent is closely related to the self-affinity concept because this exponent describes processes undergoing scaling, which is one of affine transformations.

HURST EXPONENT

The behaviour of elements in magmatic processes can be determined using fractal statistics. The fractal dimension can be calculated with application of the Hurst exponent. Originated from nonlinear dynamics and based on the fractal properties of Brownian motion, the Hurst exponent is frequently used for the detection of trends and the 'memory effect' in chaotic processes, also in magma crystallizing processes (Hoskin 2000). Quantitative analyses of variable data series, based on determination of the Hurst exponent, are successfully applied in various fields of science: natural, economic, and medical (Yang and Lo 1997; Peters 1997; West 1990).

The fractal dimension is related to the Hurst exponent (Hastings and Sugihara, 1993) according to the formula:

$$D = D_T + 1 - H$$

where: D_T is the topological dimension; D is the fractal dimension; H is the Hurst exponent.

Robert Brown, an English botanist, observed in 1827 that flower pollen dissipated in water (suspension) made a permanent chaotic motion. Brown believed that this motion was a manifestation of life. Of course, Brown was wrong but his observation led in consequence to proving that the second law of thermodynamics is of a purely statistical character with regard to average values. Brownian motion was also a premise for the existence of atoms, which could not be explained under the assumption that the structure of matter is continuous. Brownian motion is a stochastic process understood as a sequence of random variables, and is characterised by several basic values such as: average value, variance, high-order variable moments and process value probability distribution.

Following the researches of Einstein (1905) and Smoluchowski (1906), that continuous stochastic process was permanently adopted for physics. Later, thanks to Wiener (1923), it was adopted by mathematicians as the mathematical description of Brownian motion. With that, the Brownian motion process is ranked among the most important models for random processes.

Later, a more general form of Brownian motion was proposed: the fractional Brownian motion, or, in other words: fractional Gaussian noise (Mandelbrot 1997). Brownian motion is a function $B_H(t)$ and is a Gaussian process with a given parameter of $H \in (0;1)$, with its average value equal to zero, and its covariance according to the formula:

$$Cov\left(B_H(t_1); B_H(t_2)\right) = \frac{\sigma^2}{2}\left(t_1^{2H} - |t_1 - t_2|^{2H} + t_2^{2H}\right)$$

where: H − Hurst exponent; σ^2 − variance.

Einstein (1905) presented a theory explaining the basis of the Brownian motion with the proof that the average square value of particle displacement (R) [interval from the starting point to the end point] is in proportion with time (t):

$$R \approx t^{1/2}$$

The hydrologist H.E. Hurst extended the Einstein model using the re-scaled range (Hurst *et al.* 1965) and found that in general:

$$R \approx t^H \quad \text{where } H \in (0,1)$$

Thanks to this extension, the Hurst exponent, which is based on fractal properties of Brownian motion and originates from nonlinear dynamics, became a tool capable of detecting trends and the 'memory effect' in apparently chaotic processes.

Making consideration for the influence of prior variables, the Hurst exponent is sensitive to those subtleties of the stochastic process under analysis that are not detected by classic statistical methods. Therefore, the Hurst exponent facilitates the determination of an increasing or decreasing quantity of a selected element in the magma mixing processes as well as the quantitative evaluation of dynamics for this nonlinear process. With application of the notion 'deterministic chaos', a statement can be made that the magma mixing process is a dynamic chaotic process but not a random process − because the magma mixing process is generated by a deterministic dynamic process. A dynamic process is defined as a deterministic mathematical formula, which determines the system status evolution in time, and, in this case: as a function of distance to consecutive points of the profile involved.

Several methods are applied for assessment of the value of the Hurst exponent (*H*), such as the dispersion method, spectral method and autocorrelation estimators. *R/S* analysis, which was introduced by Hurst (1951), and is the outcome of his discriminating research over a long period into the 'memory effect', is ranked among the most popular methods. The *R/S* analysis method, commonly known as the Rescaled Range Method, is a method for the detection of deterministic chaos which enables demonstrating that the evolution of numerous (apparently chaotic) data series is foreseeable, and therefore not random.

For calculations, the authors applied the rescaled range determination algorithm for a series of data y_t, where $t = 1,2\ldots,n$. Calculation steps are as follows (Peters 1994):

1. Determination of average value, $M^{(n)}$, and standard deviation, $S^{(n)}$, for y_t
2. Determination of $Y_k^{(n)} = \sum_{t=1}^{k}(y_t - M^{(n)})$ for $k = 1,2, \ldots,n$
3. Calculation of range: $R^{(n)} = \max_k(y_k^{(n)}) - \min_k(y_k^{(n)})$
4. Calculation of rescaled range: $R/S_n \overset{df}{=} \frac{R^{(n)}}{S^{(n)}}$

In order to assess the Hurst exponent (H) value by application of R/S analysis, the series of data x_t, where $t = 1, 2, \ldots, n$, should be divided in subseries of n length, where n is in sequence of each of dividers for N value meeting the condition: $2 \leq n \leq N/2$ (complying with Peters' recommendations, values of $n \geq 10$ were used for calculations). For the fixed n value, $R/S_n^{(i)}$ is calculated for each separate 'i' subseries as in steps 1–4. Then, the arithmetic average $\overline{R/S_n^{(i)}}$ is calculated, and this value is adopted as the sought R/S_n value for the given n. This procedure is applied for consecutive n values. A series of values, (R/S_n) is obtained. These data are dependent on n as follows:

$$(R/S)_n = an^H$$

where:

a – a certain positive constant,
n – length of subseries for observation set,
H – Hurst exponent.

In order to calculate the Hurst exponent, a logarithmic transformation should be made and then the following formula solved:

$$Log(R/S) = Log(a) + H(Log(n))$$

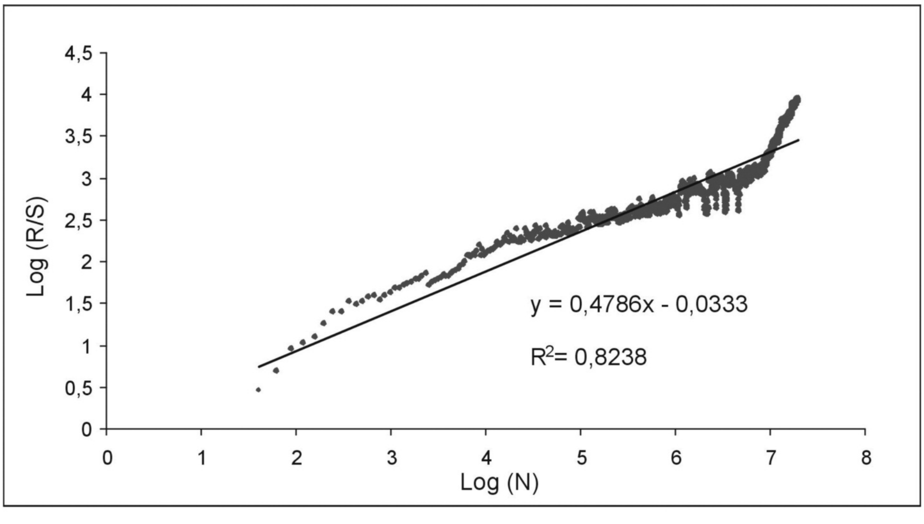
with application of the linear regression method (the least square method was applied for this case). Then, a diagram is produced with a double logarithmic scale: $Y\ axis = \log R/S$, $X\ axis = \log n$. The straight line slope is an estimation of the Hurst exponent value (Text-fig. 1).

Three classes for the Hurst exponent value exist:

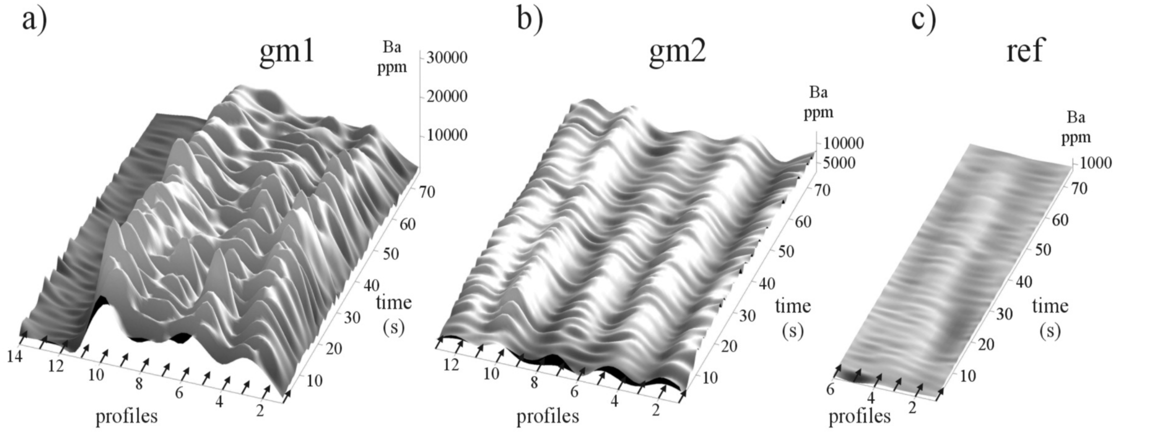
- for $0.5<H<1$: the data series is called 'persistent'. It features the long-lasting memory effect and, therefore, a high degree of positive correlation. In theory, previous data have a permanent influence on consecutive data in the series. Using notions from chaotic dynamics: a subtle sensitivity to starting conditions exists. On the other hand, using probabilistic notions: if a growth trend existed in the past for a certain time, then, for e.g. $H = 0.8$, a probability of 80% exists that this trend would be maintained in future.
- for $H = 0.5$: the system behaves randomly, being either a realization of independent variable data of monotonous distribution (Brownian motion) or a random walk. Then, its statistical structure fully executes statistic's assumptions of independence and of normality at its limit.
- for the last case of $H \in (0; 0.5)$, the system relocates for a shorter distance than, for instance, Brownian motion, and with more frequent alterations in direction. This is the case of negative correlation (anti-persistent), which means: if $H = 0.2$, the 80% probability exists that in future the process will alter its direction from the present one. Therefore, the occurrence of reduction trends or growth trends implies a trend to future alterations in direction. In practice, H values in this range are proof of a process with very high dynamics.

DESCRIPTION OF THE BEHAVIOUR OF ELEMENTS IN PHASE CRYSTALLIZATION PROCESS UNDER A MAGMA MIXING REGIME

Having applied a Hurst exponent value of between 0 and 1 to the analysis of a crystal growth process



Text-fig. 1. Graphic presentation of solution for formula for an exemplary series of data. The linear regression method was applied, and the straight line formula: $y = 0.4786x - 0.0333$ was obtained. The straight line inclination (slope) is an estimation of the Hurst exponent



Text-fig. 2. 3D-depiction of the distribution of barium concentration in three crystals (figure taken from Słaby *et al*. 2008): a – Crystal grown under intensive mixing in an active region of the magmatic field (crystal signature: gm1); b – Crystal grown under moderate progress in magma mixing, with the process being close to completion (crystal signature: gm2); c – Fragment of a crystal grown in a coherent region (crystal signature ref). Hurst exponent values were calculated for profiles indicated by black arrows. Explanation: diagram axis – X (profiles): LA ICP MS craters; Y (time) – laser pulses, each causing ablation of 5 µm thick feldspar layer; Z – Ba concentration [ppm]

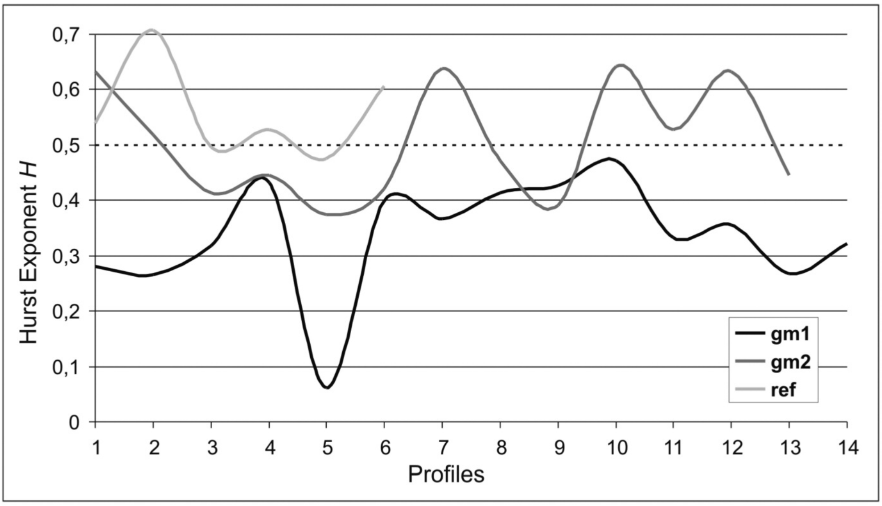
under a magma mixing regime, it can be said that low Hurst exponent values are indicators of the anti-persistent behaviour of elements and, in contrast, high values are indicators of the persistent behaviour of elements, while they are provided to the surface of the crystallizing phase. The geochemical data and the origin of the crystals, as based on these data, have been published in Słaby *et al.* (2008). The same crystals were taken for the Hurst exponent calculation presented in this paper. Text-fig. 2 (from Słaby *et al*. 2008 with modifications) shows the 3D visualization of barium distribution in a crystal formed during intensive mixing of magmas of crustal- and mantle-origin (Słaby *et al*. 2008). LA ICP MS measurements were used as data for depiction.

The depictions show crystals featuring zonal growth morphologies. Crystals growing under a dynamic magma mixing regime (Text-fig. 2a) show a growth texture specific for their micro-domain, indicating a complex pattern of mixed magma domains, and no magma homogenization. With progressing homogenization, the distribution of elements becomes more regular (Text-fig. 2b). The depiction of barium distribution reveals a regular zonal pattern. If the crystallization environment is a coherent field, the crystal is homogeneous (Text-fig. 2c). The Hurst exponent was calculated for those crystals along the indicated profiles in order to analyse the element behaviour in the crystallization process (Text-fig. 2). The calculated values are shown in Table 1 and Text-fig. 3.

In a gm1 zone crystal, the Hurst exponent value (*H*) varies between 0.06 and 0.47 for the chosen profiles. The Hurst value points to chaotic behaviour of elements during intensive mixing of magmas. The chaotic stretching and folding process of magma domains is in full progress; the degree of homogenization is low. During crystallization, the behaviour of elements is strongly anti-persistent. For zones within a *H>0.5* crystal, the process of element incorporation goes over a longer path than a random walk and shows strengthened tendencies for durability. Those zones crystallized in domains located close to coherent composition of the magma field, or in active domains featuring a high homogenization (crystals gm2, ref).

LA ICP MS spots		*profiles*													
		1	2	3	4	5	6	7	8	9	10	11	12	13	14
gm1	***H***	0.280	0.266	0.317	0.431	0.061	0.399	0.366	0.413	0.427	0.471	0.333	0.355	0.267	0.322
gm2		0.631	0.518	0.414	0.445	0.373	0.421	0.637	0.471	0.390	0.642	0.527	0.631	0.444	
ref		0.538	0.707	0.496	0.528	0.474	0.605								
gm1	***D***	1.720	1.734	1.683	1.569	1.939	1.601	1.634	1.587	1.573	1.529	1.667	1.645	1.733	1.678
gm2		1.369	1.482	1.586	1.555	1.627	1.579	1.363	1.529	1.610	1.358	1.473	1.369	1.556	
ref		1.462	1.293	1.504	1.472	1.526	1.395								

Table 1. Values of the Hurst exponent (*H*) and fractal dimension (*D*) for profiles for crystals



Text-fig. 3. Diagram of fractal dimension values (*D*) for each profile for crystals: gm1, gm2, ref

CONCLUSIONS

Fractal statistics is a very sensitive tool, which perfectly shows increasing or decreasing dynamics in a system tending to homogenize or remaining far from homogenization. In crystals, fractal statistics provides the possibility of determining which domain in a growing feldspar could be assigned to the active region, and which to the coherent region of a magma field. The behaviour of elements during the magma mixing process is anti-persistent. Even when the system proceeds toward homogenization, and the anti-persistent character is weakened, it nevertheless still remains. This means that, even for crystals featuring an absence of a clear zonal structure, the grade of element behaviour persistence facilitates identifying the process as representing nonlinear dynamics, which is an indication of magma mixing. The Hurst exponent is a perfect tool for describing those processes. Crystallization from coherent magma domains changes the element behaviour from anti-persistent to persistent. Therefore, if a process featuring nonlinear dynamics is overlapped by another process of a different character, fractal statistics seems to be the ideal tool to separate those processes. Thus, the Hurst exponent can be used for two operations: as a tool for describing nonlinear processes, e.g. differentiation of the magma field composition, and for separation of magma- from post-magma- process effects – if elements show a different behaviour in each of the above-mentioned processes.

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